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RADAR AND SONAR AMBIGUITY
FUNCTIONS AND GROUP THEORY

by

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Abstract

Group theory is applied to unify wideband radar theory with narrowband radar theory through contractions. Following Kalnins and Miller in their work with contractions, we correct their calculations, completing their theory. We also unify different forms of the wideband ambiguity function in the engineering literature through group theory and begin some work on some discrete transforms, the finite counterpart of wideband and narrowband theory.

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Glossary

a	dilation factor
A_i	The i th basis vector for \mathcal{G}_A
$*$	A group operation symbol In appendix, cyclic convolution
A_i	The i th affine group
A_p	The affine group with coefficients in \mathcal{Z}_p (in appendix only)
b	The time delay in the wideband case
BW	The Bandwidth of the signal
\bullet	The alternate group operation symbol
c	The speed of the signal through the medium
$C_{e,s}$	The correlation between two signals
C_i^ϵ	The i th member of basis of the ϵ -groups corresponding to G_A
$C_{U,J,s}$	The coefficient of the representation U
$DAFT$	The Discrete Affine Fourier Transform
DFT	The Discrete Fourier Transform
ϵ	The parameter used in contractions
$e_n(t)$	The narrowband echo
$e_w(t)$	The wideband echo
\mathcal{F}_S	The vector space of all complex functions on S

G	A group
\hat{G}	The dual group of G
G_A	The general affine group
\mathcal{G}_A^c	The Lie algebra of G_A^c
\mathcal{G}_G	The Lie algebra of a group G
$GL(U)$	General linear group of the the vector space V
\mathcal{H}	The Heisenberg group
<i>IDAFT</i>	The Inverse Affine Fourier Transform
<i>IDFT</i>	The Inverse Discrete Fourier Transform
$K^\lambda(a, b, c)$	The representation of G_A used in contractions
$L^2(\mathcal{R})$	The Hilbert space of square integrable functions
$N_{f,g}(x, y)$	The narrowband ambiguity function
ω	The Doppler frequency shift
ω	The primitive third root of unity (in appendix only)
$\phi_{i,j}$	The isomorphism from \mathcal{A}_i to \mathcal{A}_j
R	The range of the object with respect to the transmitter
$s(t)$	The transmitted signal
τ	The time delay in the narrowband case
T	The time duration of the signal
U_G	The unitary representation of a group G
$U^\lambda(a, b, c)$	The unitary representaton of G_A

V_N	Narrowband ambiguity volume
V_W	Wideband ambiguity volume
$W_{f,g}(a, b)$	The wideband ambiguity function
$W_{f,g}^i(a, b)$	The i th ambiguity function corresponding to \mathcal{A}_i
\mathcal{Z}_n	The group of integers modulo n

Chapter 1

Introduction

Narrowband and wideband ambiguity functions are fundamental in the study of parameter estimation by active radar and sonar systems. Interest in ambiguity function theory has been renewed due to the current intense research activity in affine wavelet transforms and other group theoretic transforms. We have attempted to continue the program of unifying scattered results in radar and sonar ambiguity theory via group representation theory. The underlying idea is that both the narrowband and wideband ambiguity functions are coefficients of the unitary representations of their respective groups. Wideband ambiguity functions are coefficients of the affine group and narrowband ambiguity functions are coefficients of the Heisenberg group. This fact provides insight into important concepts of admissible signals, ambiguity conservation, and ambiguity function invariance properties that are important for signal design.

The history of the applications of group theory to radar and sonar is not long since the subject has been less than fifty years old. The stage opened in 1953 with P. M. Woodward's seminal book, *Probability and Information Theory, with Applications to Radar* [44, pages 115-125], in which he detailed the radar ambiguity function and its uses. Volume conservation was noted, and the chapter closed with

the problem of signal design, based on the desired radar ambiguity function shape: *the radar synthesis problem*. Wilcox [43] attacked the problem via the techniques of mathematical analysis with some success in 1960, but it was not until around 1983 that Walter Schempp started publishing a series of papers culminating in his article [39] and book [38]. These papers detailed the connection of the radar ambiguity function with the Heisenberg group and attacked the problem from the viewpoint that the radar autoambiguity function is a positive definite function on its group. A lot was known about positive definite functions already, especially through the work of Naimark and Gelfand, summarized in the former author's work [31]. In 1985, L. Auslander and R. Tolimieri [4] took Wilcox's aforementioned paper and studied it from the group theoretical viewpoint [4] (as suggested by Schempp, since they have cited one of his preprints?). Since the Heisenberg group had a discrete subgroup, they could apply nilmanifold theory, a theory they had written about extensively earlier [3].

About 1985, Sibul and Titlebaum started investigating the wideband ambiguity function [37], following the pioneering work by R. Altes. In 1987, Sibul with the help of R. Urwin wrote a proposal for funding in investigating the role of group theory with signal processing. After finding that the affine group was the fundamental concept for the wideband ambiguity function, we found that Auslander and Gertner detailed the same idea and more in [2]. Furthermore, Auslander and Gertner described the approximation of the wideband to the narrowband in such a

clear way that we began to wonder if there was any connection between the affine and the Heisenberg groups. By this time, A. Banyaga joined us as mathematical advisor. In 1989, we decided that one had to be a contraction of another. But within minutes after our meeting, Banyaga found a preprint of Kalnins and Miller [26] in the lounge showing that our suspicion was indeed right. However, we did find some major conceptual errors, although the basic idea is correct. Their article has been published, still with those errors. So part of this thesis is to correct them.

As this thesis was being written, there were ongoing questions in the signal processing community. One of those questions was concerning the existence of at least two forms of wideband ambiguity functions and when and how to use them properly. We found that these ambiguity functions arose from different forms of the affine group, and so have included the result.

Finally we include some results of a another investigation in the applications of Group theory. We wanted to follow the methodology of Auslander and Tolimieri [3] in their treatment of the FFT to create a transform based on the finite affine group.

A brief overview of the thesis now follows: Chapter 2 will sketch some ideas concerning the narrowband and wideband ambiguity function. Chapter 3 then reviews some mathematics necessary for the rest of the thesis and then shows how the ambiguity functions are related to group theory. We conclude the chapter with the unification of the various forms of the wideband ambiguity in the engineering

literature. Chapter 4 shows how the wideband ambiguity function may approximate the narrowband under suitable conditions, and then shows how contractions may explain the phenomenon. Chapter 5 then applies a celebrated theorem to ambiguity volume and then shows how ambiguity volume depends on contractions. Finally, our results on the finite affine group are stated in the appendix.

Our original contributions consist of corrections of Kalnins and Miller's results concerning contractions. The second is the unifying different forms of the wideband ambiguity function in the engineering literature. The third is the investigation of ambiguity volume with contractions. And the final is an explicit construction of the group theoretic transform based on the finite affine group.

Chapter 2

The Ambiguity Functions

2.1 Introduction

This chapter provides a sketch of ambiguity function theory. It is not intended to be exhaustive, but to be a motivation for the rest of the thesis. A good reference for further information is [12, 30, 45].

2.2 Narrowband and Wideband Echoes

Radar and active sonar use the same principle. The systems consist of a *transmitter/receiver* with a processing unit. A waveform, called a *signal* is transmitted towards an object of interest, such as an airplane preparing to land on the runway. After a time delay, the transmitter/receiver receives a reflected form of the signal, called the *echo*, from which the processing system may extract important information such as the position and radial velocity of the object. Radar uses electromagnetic waves in the atmosphere, while active sonar uses acoustic waves in water.

It is assumed that the environment of the system is free of clutter. The waveform travels from the system to the object and back without any interference. Furthermore, we assume the object to be a *point*. The reflected waveform will be

identical to the transmitted waveform, delayed of course, if the object were not moving. Finally we assume the object to be travelling at a constant radial velocity, v . The ramifications of this assumption will be seen in section 4.2.

Finally, a signal $s(t)$ may be seen essentially as an element of $L^2(\mathcal{R})$. It has time as its domain and voltage as its range. The signal's Fourier transform may be interpreted as the set of its frequencies, or *spectrum*.

The three assumptions above provide a basis for radar/sonar theory in general. Suppose that the signal was *narrowband* as well. Narrowband signals are those whose Fourier transforms have very small support, called the *bandwidth*(BW), so that they essentially appear to be impulse functions in the frequency domain. This also means that their Fourier transforms are concentrated about a central frequency. We also assume that the signal has a very short time duration T . The narrowband assumption is coupled with the time duration assumption as follows [41, page 241]:

$$\frac{2|v|}{c} \ll \frac{1}{T \cdot BW}, \quad (2.1)$$

where c denotes the speed of the signal in the medium. Under this additional assumption, the echo takes the form

$$e_n(t) = s(t + \tau)e^{i\omega t}. \quad (2.2)$$

Here, τ denotes the time elapsed between the transmission of the signal and re-

ception of the echo, called the *time delay*, from which the position relative to the transmitter/receiver may be calculated. Let c denote the speed of the signal in the medium. Then the position R of the object is calculated as:

$$R = \frac{c\tau}{2}, \quad (2.3)$$

since we assumed that the object is travelling at speeds much less than c . We divide by 2, taking account that the signal had to travel to the object and back.

The symbol ω denotes the *doppler shift* and basically is related to the radial velocity of the object. For example, the pitch of a car moving towards an observer will be perceived higher than when it is moving away. The radial velocity of the object given the doppler shift may be calculated using the formula:

$$\omega_0 \cdot \frac{2v}{c} = \omega, \quad (2.4)$$

where ω_0 is the essential frequency of the transmitted signal.

Thus we may process the return signal to find the object's radial velocity and range.

Without the additional narrowband assumption, the return signal takes the form:

$$e_w(t) = \sqrt{a}s(at + b) \quad (2.5)$$

where b is related to the delay of the first transmitted photon and

$$a = \frac{1 + \beta}{1 - \beta} \quad (2.6)$$

where $\beta = \frac{v}{c}$. Note that now the time scale of the return signal is dilated or contracted depending on whether the object is moving away or towards our transmitter/receiver. In fact, nature is wideband, meaning that the narrowband formulation is an approximation. We will explore this further in section 4.2.

2.2.1 The Ambiguity Functions

A final assumption is that a correlation receiver produces forms a sufficient statistic for detection in real white Gaussian noise [40, page 269]. The location and velocity of the object may be determined by “comparing” the echo with shifts of the transmitted signal. The maximum value corresponds to the location and velocity of the object.

The correlation between two signals $f(t)$ and $g(t)$ is:

$$\int_{-\infty}^{\infty} f(t) \overline{g(t)} dt, \quad (2.7)$$

recognizable as the inner product of $L^2(\mathcal{R})$.

Basically, to determine the location and velocity of the object, one simply correlates the echo with shifts of the transmitted signal:

$$C_{e,s}(a, \tau) = \int_{-\infty}^{\infty} e(t) \overline{s(at + \tau)} dt. \quad (2.8)$$

The parameters (a, τ) that maximize the value of $C_{e,s}$ are those which best models the received signal. The location and speed of the target can be readily calculated. Each correlation process is similar for each type of echo.

Abstracting our correlation process, we obtain the narrowband and the wideband ambiguity functions $N_{f,g}$ and $W_{f,g}$:

$$N_{f,g}(x, y) = \int_{-\infty}^{\infty} f(t) \overline{g(t + x)} e^{-2\pi i y t} dt \quad (2.9)$$

$$W_{f,g}(a, b) = \int_{-\infty}^{\infty} f(t) \overline{g(at + b)} dt \quad (2.10)$$

These two functions are the key to many synthesis and design problems in radar.

Chapter 3

Group Theory and Radar and Sonar

3.1 Introduction

After introducing the signal ambiguity functions in the previous chapter, we are now in a position to see how group theory provides a unified view. We review some mathematical background, with special emphasis on relevant techniques, and then show how signal ambiguity functions arise as connected to special groups. Finally, we unify the different forms of the wideband ambiguity functions.

3.2 Some Mathematical Background

Basic knowledge of group theory and differentiable manifolds, along with the theory of Lie groups and Lie algebras will be assumed. We also use knowledge of elementary Hilbert space freely. However, for clarity, we will review the concepts of group representation theory and also the little known topic of parametrization.

The set of operators on a Hilbert space that preserve its inner product forms the *unitary group* under the operation of composition. Specifically, the space $L^2(\mathcal{R})$ will be our Hilbert space under consideration.

It is possible to define measures on topological groups and hence Lie groups, such as the Heisenberg group and the affine group. These measures are compatible

with the group structure in the sense that if a set is translated by multiplication of a group element, its measure remains unchanged. There can be *left invariant* and *right invariant* measures, depending on whether the set is multiplied by the group element on the left or on the right. Explicitly, if f is a measurable function on a group G , the measure μ is left-invariant, if

$$\int_G f(g_0 * g) d\mu(g) = \int_G f(g) d\mu(g), \quad (3.1)$$

for all $g_0 \in G$. The right invariant measure is defined similarly. If an invariant measure, left or right, exists for a group, it is well known that it must necessarily be unique up to multiplication by a constant. Finally, if the left and right invariant measure coincide, the group in question is said to be *unimodular*.

A *unitary representation* U of a group G on a Hilbert space H is a mapping assigning to each group element a unitary operator $U(x)$ on H , such that for any $x, y \in G$,

$$U(xy) = U(x) \circ U(y) \quad (3.2)$$

(Recall that U is a unitary operator on H if, for all $v, w \in H$, $\langle Uv, Uw \rangle = \langle v, w \rangle$).

A subspace S of H is *invariant* under U if for any $v \in S$, $U(x) \cdot v \in S$ for all $x \in G$. A representation U is said to be *irreducible* if the only invariant subspaces of H are the zero subspace and H itself.

$$\begin{array}{ccccc}
 G' & \times & G' & \longrightarrow & G' \\
 \downarrow & & \psi \times \psi & & \downarrow \psi \\
 G & \times & G & \longrightarrow & G
 \end{array}$$

Figure 3.1: Calculation of new group operation under new coordinates.

The *coefficient* of a continuous unitary group representation U of G on a Hilbert space H with respect to the ordered pair (f, g) , both f and g in H , is a mapping $C_{U,f,g} : G \longrightarrow \mathcal{R}$:

$$x \longmapsto \langle f, U(x)g \rangle . \quad (3.3)$$

The coefficient is a generalization of the usual Fourier coefficients.

Finally, we cover the area of coordinates. Given a group, with an operation, a one-to-one transformation ψ from the group *onto* itself will change its operation. Referring to figure refcomdiagram, the group G and G' are the group with the original operation $*$ and the new group with the new operation \bullet respectively. Beginning with $*$, we may calculate \bullet by simply noting that:

$$\bullet = \psi \circ * \circ (\psi \times \psi)^{-1} \quad (3.4)$$

This technique is essential for the correct contractions on the group level, as will be shown later. Corwin and Greenleaf [13, pages 14-16] is one of the few references detailing this technique.

3.3 The Affine and Heisenberg Groups

The subgroup of $Gl(2, \mathbf{R})$ consisting of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, with $a > 0$ and b any real number, is called the *real (two dimensional) affine group*. It will be denoted by \mathcal{A}_1 . Its identity element is the usual identity element of $Gl(2, \mathbf{R})$, and the inverse of an element $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix}$. Sometimes we may denote an element $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ by just (a, b) . We may thus see the affine group as the set $\mathbf{R}_+ \times \mathbf{R}$ with the operation:

$$(a, b) * (a', b') = (aa', ab' + b). \quad (3.5)$$

We also will need to work with the affine group cross the real line, $G_{\mathcal{A}}$, the group of matrices of the form:

$$\begin{pmatrix} a & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{pmatrix}, \quad (3.6)$$

where again $a > 0$, b and x are any real number.

Another matrix group of particular interest is the so-called *Heisenberg group*, \mathcal{H} , consisting of matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.7)$$

Clearly, the identity and the inverse of this set belongs to \mathcal{H} . Here, \mathcal{H} is a three dimensional manifold. Alternately, we may see \mathcal{H} , as \mathbf{R}^3 with the operation

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy'). \quad (3.8)$$

However, \mathcal{H} has another form, if one lets the ψ of the equation 3.4 be

$$(x, y, z) \longmapsto (x, y, z - \frac{1}{2}xy), \quad (3.9)$$

we obtain a new multiplication rule:

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)), \quad (3.10)$$

which is called the *symplectic* parametrization of \mathcal{H} . Needless to say, there are many other forms of the Heisenberg group.

Both groups play a fundamental role in signal processing via their unitary representations on $L^2(\mathcal{R})$.

Proposition 3.3.1 *The affine and Heisenberg groups have unitary representations on $L^2(\mathcal{R})$ as follows:*

$$a. U_{\mathcal{A}_1} : \mathcal{A}_1 \rightarrow \mathcal{U}(L^2(\mathcal{R}))$$

$$U_{\mathcal{A}_1}(a, b)f(t) = \frac{1}{\sqrt{a}}f\left(\frac{t-b}{a}\right) \quad (3.11)$$

$$b. U_{G_{\mathcal{A}}} : G_{\mathcal{A}} \rightarrow \mathcal{U}(L^2(\mathcal{R}))$$

$$U_{G_{\mathcal{A}}}(a, b, x)f(t) = \frac{1}{\sqrt{a}}e^{2\pi i x}f\left(\frac{t-b}{a}\right) \quad (3.12)$$

$$c. U_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{U}(\mathcal{L}^{\epsilon}(\mathcal{R}))$$

$$U_{\mathcal{H}}(x, y, z)f(t) = e^{2\pi i z}e^{2\pi i y t}f(t+x) \quad (3.13)$$

Proof

The proof consists in verifying the definition of group representations:

Proof of a.:

$$U_{\mathcal{A}_1}(a, b)U_{\mathcal{A}_1}(a', b')f(t) = U_{\mathcal{A}_1}(a, b)\left[\frac{1}{\sqrt{a'}}f\left(\frac{t-b'}{a'}\right)\right] \quad (3.14)$$

$$= \frac{1}{\sqrt{aa'}}f\left(\frac{\frac{t-b}{a}-b'}{a'}\right) \quad (3.15)$$

$$= \frac{1}{\sqrt{aa'}}f\left(\frac{t-ab'-b}{aa'}\right) \quad (3.16)$$

$$= U_{\mathcal{A}_1}(aa', ab'+b)f(t). \quad (3.17)$$

Proof of b.:

$$U_{G_A}(a, b, x)U_{G_A}(a', b', x')f(t) = U_{G_A}(a, b, x) \left(\frac{1}{\sqrt{a'}} e^{2\pi i x'} f\left(\frac{t-b'}{a'}\right) \right) \quad (3.18)$$

$$= \frac{1}{\sqrt{aa'}} e^{2\pi i x} e^{2\pi i x'} f\left(\frac{\frac{t-b}{a} - b'}{a'}\right) \quad (3.19)$$

$$= \frac{1}{\sqrt{aa'}} e^{2\pi i(x+x')} f\left(\frac{t-ab'-b}{aa'}\right) \quad (3.20)$$

$$= U_{G_A}(aa', ab'+b, x+x')f(t). \quad (3.21)$$

Proof of c.:

$$U_{\mathcal{H}}(x, y, z)U_{\mathcal{H}}(x', y', z')f(t) = U_{\mathcal{H}}(x, y, z)e^{2\pi i z'} e^{2\pi i y' t} f(t+x') \quad (3.22)$$

$$= e^{2\pi i z} e^{2\pi i y t} e^{2\pi i z'} e^{2\pi i y'(t+x)} f(t+x+x') \quad (3.23)$$

$$= e^{2\pi i(z+z'+xy')} e^{2\pi i(y+y')t} f(t+x+x') \quad (3.24)$$

$$= U_{\mathcal{H}}(x+x', y+y', z+z'+xy')f(t). \quad (3.25)$$

We may now see that the wideband and narrowband ambiguity functions are simply the coefficients of the affine and Heisenberg groups respectively.

But $W_{f,g}$ does not strictly look like an ambiguity function. The next section will help clarify this.

Hence we may use group theory as an approach in analyzing the behavior of ambiguity functions. In fact, Schempp and others studied the autoambiguity

functions through this view point, since such functions are positive definite functions on the groups, and many results concerning these were available through the efforts of Gelfand and Naimark [31].

3.4 Various Forms of the Wideband Ambiguity Function Unified

The form of the wideband ambiguity function above appears in the guise of wavelets as well in works by Daubechies and Miller: we shall call it W^1 :

$$W^1(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{g\left(\frac{t-b}{a}\right)} dt. \quad (3.26)$$

Speiser and others deal with a second form of the wideband ambiguity function:

$$W^2(a, b) = \sqrt{a} \int_{-\infty}^{\infty} f(t) \overline{g(at + b)} dt. \quad (3.27)$$

This is the "usual" wideband ambiguity function. The third form:

$$W^3(x, y) = e^{-\frac{x}{2}} \int_{-\infty}^{\infty} f(t) \overline{g(e^{-x}t - y)} dt \quad (3.28)$$

appears in Heil and Walnut, and subsequently in the thesis by Fowler. And the fourth form, which is close to the heart of engineers in signal processing, initially appeared in Kelly-Wishner and Altes:

$$W^4(a, b) = \sqrt{a} \int_{-\infty}^{\infty} f(t) \overline{g[a(t + b)]} dt. \quad (3.29)$$

We would like to show that all these forms are coefficients of the affine group, albeit with different parametrizations. To this purpose, we propose the definitions of four groups:

Definition 3.4.1 *Let the four groups be defined as follows:*

- a. *The matrix group \mathcal{A}_1 consisting of the set of matrices of the form*

$$\begin{pmatrix} s & \tau \\ 0 & 1 \end{pmatrix} \text{ with real entries and } s > 0.$$
- b. *The matrix group \mathcal{A}_2 consisting of the set of matrices of the form*

$$\begin{pmatrix} s & 0 \\ \tau & 1 \end{pmatrix} \text{ with real entries and } s > 0.$$
- c. *The set \mathbf{R}^2 with the operation:*

$$(x, y) * (x', y') = (x + x', e^{-x'}y + y') \quad (3.30)$$

is the group \mathcal{A}_3 .

- d. *The set $\mathbf{R}_+ \times \mathbf{R}$ with the operation*

$$(a, b) * (a', b') = (aa', b + \frac{b'}{a}) \quad (3.31)$$

is the group \mathcal{A}_4 .

Proposition 3.4.1 *\mathcal{A}_{1-4} are all isomorphic.*

Proof. 1. The group \mathcal{A}_1 is isomorphic to \mathcal{A}_2 , by the isomorphism which sends a matrix to its inverse transpose. Indeed, consider $\phi_{1,2} : \mathcal{A}_1 \rightarrow \mathcal{A}_2$, where

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{a} & 0 \\ -\frac{b}{a} & 1 \end{pmatrix}. \quad (3.32)$$

Then

$$\phi_{1,2}\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix}\right) = \phi_{1,2}\left[\begin{pmatrix} aa' & ab' + b \\ 0 & 1 \end{pmatrix}\right] \quad (3.33)$$

$$= \begin{pmatrix} \frac{1}{aa'} & 0 \\ -\frac{ab'+b}{aa'} & 1 \end{pmatrix} \quad (3.34)$$

$$= \begin{pmatrix} \frac{1}{a} & 0 \\ -\frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a'} & 0 \\ -\frac{b'}{a'} & 1 \end{pmatrix} \quad (3.35)$$

$$= \phi_{1,2}\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix}\right) \quad (3.36)$$

The homomorphism $\phi_{1,2}$ is clearly one-to-one and onto.

2. The group \mathcal{A}_2 is isomorphic to \mathcal{A}_3 by the isomorphism $\phi_{2,3} : \mathcal{A}_2 \rightarrow \mathcal{A}_3$ where $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mapsto (-\ln a, b)$, and changing variables $\ln a \mapsto a$. Indeed,

$$\phi_{2,3}\left\{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix}\right\} = \phi_{2,3}\left\{\begin{pmatrix} aa' & 0 \\ a'b + b' & 1 \end{pmatrix}\right\} \quad (3.37)$$

$$= (-\ln(aa'), a'b + b') \quad (3.38)$$

$$= (-\ln a - \ln a', e^{\ln a'} b + b') \quad (3.39)$$

$$= (-\ln a, b)(-\ln a', b') \quad (3.40)$$

$$= \phi_{2,3}\left\{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}\right\}\phi_{2,3}\left\{\begin{pmatrix} a' & 0 \\ b' & 1 \end{pmatrix}\right\} \quad (3.41)$$

The map $\phi_{2,3}$ is certainly one-to-one and onto. Now, we just change variables: let $x = -\ln a$ and $x' = -\ln a'$. Hence:

$$(-\ln a, b)(-\ln a', b') = (-\ln a - \ln a', a'b + b') \quad (3.42)$$

becomes:

$$(x, b)(x', b') = (x + x', e^{-x'}b + b'), \quad (3.43)$$

the rule of \mathcal{A}_3 .

3. The group \mathcal{A}_2 is isomorphic to \mathcal{A}_4 by the isomorphism: $(a, b) \mapsto (a, ab)$.

Indeed,

$$\phi_{2,4}(a, b)\phi_{2,4}(a', b') = (a, ab)(a', a'b') \quad (3.44)$$

$$= (aa', a'ab + a'b') \quad (3.45)$$

$$= (aa', a'(ab + b')) \quad (3.46)$$

$$= \phi_{2,4}(aa', a'(ab + b')) \quad (3.47)$$

Proposition 3.4.2 *The following maps are representations of \mathcal{A}_1 through \mathcal{A}_4 on $L^2(\mathcal{R})$ respectively:*

$$U_1(a, b)f(t) = \frac{1}{\sqrt{a}}f\left(\frac{t-b}{a}\right) \quad (3.48)$$

$$U_2(a, b)f(t) = \sqrt{a}f(at + b) \quad (3.49)$$

$$U_3(x, y)f(t) = e^{-\frac{x}{2}}f(e^{-x}t - y) \quad (3.50)$$

$$U_4(x, y)f(t) = \sqrt{a}f(a[t + b]) \quad (3.51)$$

Proof The proof consists of simple verification, as follows:

- a. The verification for U_1 : done in proposition 3.3
- b. The verification that U_2 is a representation of \mathcal{A}_2 is as follows:

$$U_2(a, b)U_2(a', b')f(t) = \sqrt{a'}U_2(a, b)f(a't + b') \quad (3.52)$$

$$= \sqrt{aa'}f(a'(at + b) + b') \quad (3.53)$$

$$= \sqrt{aa'}f(aa't + a'b + b') \quad (3.54)$$

$$= U_2(aa', a'b + b')f(t) \quad (3.55)$$

- c. The verification that U_3 is a representation of \mathcal{A}_3 is as follows:

$$U_3(a, b)U_3(a', b')f(t) = U_3(a, b)e^{-\frac{a}{2}}f(e^{-a'}t - b) \quad (3.56)$$

$$= e^{-\frac{a+a'}{2}}f(e^{-a'}(e^{-a}t - b) - b) \quad (3.57)$$

$$= e^{-\frac{a+a'}{2}}f(e^{-a+a'}t - e^{-a'}b - b') \quad (3.58)$$

$$= U_3(a + a', e^{-a'}b + b')f(t) \quad (3.59)$$

d. The verification that U_4 is a representation of \mathcal{A}_4 is as follows:

$$U_4(a, b)U_4(a', b')f(t) = U_4(a, b)\sqrt{a'}f[a'(t + b')] \quad (3.60)$$

$$= \sqrt{aa'}f[a'\{a(t + b) + b'\}] \quad (3.61)$$

$$= \sqrt{aa'}f[a'at + a'ab + a'b'] \quad (3.62)$$

$$= \sqrt{aa'}f\left[aa'\left(t + \frac{aa'b + a'b'}{aa'}\right)\right] \quad (3.63)$$

$$= U_4(aa', b + \frac{b'}{a})f(t) \quad (3.64)$$

Hence, forming coefficients as usual, we have all the wideband forms in the engineering literature (that we know of).

3.5 Invariance Properties of the Ambiguity Functions

We now apply the concepts of the previous sections to unify other results. In [1], we find various invariance properties of the wideband ambiguity function as detailed by Altes. He used the formula:

$$W_{f,g}(a,b) = \int_{-\infty}^{\infty} f(t) \overline{g[a(t+b)]} dt, \quad (3.65)$$

and so we will reprove some of his statements using the group theoretic viewpoint.

The key idea is that a unitary representation of a group G preserves inner products:

$$\langle U(x)f, U(x)g \rangle = \langle f, g \rangle. \quad (3.66)$$

The trick is to be careful to keep track of which form one is using as to which group one is working with.

Proposition 3.5.1

$$W_{f,g}^4(a,b) = \overline{W_{g,f}^4(1/a, -ab)}. \quad (3.67)$$

Proof

$$W_{f,g}^4(a,b) = \langle f, U_4(a,b)g \rangle \quad (3.68)$$

$$= \langle U_4(a,b)^{-1}f, U_4(a,b)^{-1}U_4(a,b)g \rangle \quad (3.69)$$

$$= \langle U_4(1/a, -b/a)f, g \rangle \quad (3.70)$$

$$= \overline{\langle g, U_4(1/a, -ab)f \rangle} \quad (3.71)$$

$$= \overline{W_{g,f}^4(1/a, -ab)}. \quad (3.72)$$

What we have done is see complicated integrals as coefficients of representations of affine groups. We emphasize that the above theorem is true for only U_4 as the inverse of an element of another group can take a different form. For example,

$$W_{f,g}^1(a, b) = \overline{W_{g,f}^1(1/a, -b/a)}. \quad (3.73)$$

Some other identities are special cases of the following proposition:

Proposition 3.5.2

$$\langle U(x)f, U(y)g \rangle = \langle f, U^{-1}(x)U(y)g \rangle. \quad (3.74)$$

Proof This proposition is a consequence of the unitary property of the representation.

For example, in the case of the group \mathcal{A}_1 , we obtain the results:

$$\langle U_1(a, b)f, U_1(a', b')g \rangle = \langle f, U_1\left(\frac{a}{a'}, \frac{b' - b}{a}\right)g \rangle, \quad (3.75)$$

which may be verified by the technique above. Thus group theory provides a shorthand notation to bypass many integration substitutions.

Studying equation 3.75 further, we find that some of Altes' identities arise.

For example,

Proposition 3.5.3 *If $f(t) = a^{\frac{1}{2}}u(at)$ and $g(t) = a^{\frac{1}{2}}v(at)$, then:*

$$W_{f,g}^1(a, b) = W_{u,v}^1(a, b). \quad (3.76)$$

Proof

We note that the hypothesis implies that $f = U_1(\frac{1}{k}, 0)u$ and that $g = U_1(\frac{1}{k}, 0)v$. Hence:

$$W_{f,g}^1(a, b) = \langle f, U_1(a, b)g \rangle \quad (3.77)$$

$$= \langle U_1(\frac{1}{k}, 0)u, U_1(a, b)U_1(\frac{1}{k}, 0)v \rangle \quad (3.78)$$

$$= \langle U_1(\frac{1}{k}, 0)u, U_1(\frac{a}{k}, b)v \rangle \quad (3.79)$$

$$= \langle u, U_1^{-1}(\frac{1}{k}, 0)U_1(\frac{a}{k}, b)v \rangle \quad (3.80)$$

$$= \langle u, U_1(a, kb)v \rangle \quad (3.81)$$

$$= W_{u,v}^1(a, kb) \quad (3.82)$$

The equation 3.80 is the result of applying proposition 3.5.2. Again, we must emphasize that the form of the equation will vary depending on the group.

Chapter 4

Contractions

4.1 Introduction

Contractions form the topic of this chapter. After motivating the idea of contraction, we apply it to the affine group.

Contractions were presented for the first time by Wigner and Inonu [42] in 1953, to try to connect special relativity with Galilean relativity and to connect classical mechanics with quantum. Not long after, Saletan [35] provided a solid foundation for Inonu and Wigner. Various papers were then published on contractions of specific groups, and soon all the contractions of dimension three or less were known [11]. In fact, the contraction we are going to be dealing with was listed in that article [11]. We follow in particular the contraction definitions of Dooley [15], since they are quite clear and sufficiently rigorous for our purposes. This chapter contains corrections to Kalnins and Miller's paper [26]. Their errors lay primarily in the improper parametrization in the Lie group level and exponentiating the Lie algebra under the assumption that it was a commutative Lie algebra, which it is not. After corrections, the contraction computations flow beautifully.

4.2 Physical Motivation

Sibul and Titlebaum [37], among others, found that under suitable conditions, the wideband ambiguity function may approximate the narrowband. The essential facts are as follows:

Let us consider the wideband ambiguity function:

$$W_{f,g}(s, \tau) = \sqrt{s} \int_{-\infty}^{\infty} f(t) \overline{g(st + \tau)} dt. \quad (4.1)$$

Now let us also assume that $g(t)$ is a finite energy signal with envelope $u(t)$ and carrier frequency ω_0 :

$$g(t) = u(t)e^{i\omega_0 t} \quad (4.2)$$

Hence the received signal would have the form of time delay and dilation:

$$g(st + \tau) = u(st + \tau)e^{i\omega_0(st + \tau)} \quad (4.3)$$

Now recall that $s = \frac{1+\beta}{1-\beta}$, where $\beta = \frac{v}{c}$. Expanding in Taylor series:

$$s = 1 + 2\beta + 2\beta^2 + \dots \quad (4.4)$$

Now suppose $v \ll c$ so that $\beta \ll 1$ and $\omega_0 \beta^2 \ll 1$, then the received signal becomes approximately:

$$g(st + \tau) \approx u(t + \tau)e^{i\omega_0(t+\tau)}e^{-i\omega_0\beta t} \quad (4.5)$$

$$= g(t + \tau)e^{-i\omega_0\beta t} \quad (4.6)$$

The associated ambiguity function becomes:

$$W_{f,g}(s, \tau) = \int_{-\infty}^{\infty} f(t)e^{i\omega_0\tau} \overline{g(t + \tau)e^{i\omega_0 t}} dt, \quad (4.7)$$

which is the narrowband ambiguity function up to a phase factor. It is interesting that this approximation by physical considerations has a mathematical counterpart. Essentially, the physical argument has shown that the coefficient of the affine group may approximate that of the Heisenberg group. But the key idea is that the affine group may approximate the Heisenberg group in a certain sense. This is essentially the concept of contractions.

4.3 Contractions

4.3.1 Definitions

Definition 4.3.1 *Let $\mathcal{G}, \mathcal{G}_1$ be two Lie algebras with the same underlying vector space U . Then \mathcal{G}_1 is a contraction of \mathcal{G} , if there exists a continuous mapping $\phi : (0, 1] \longrightarrow \mathcal{GL}(U)$ such that*

$$\lim_{\lambda \rightarrow 0} \phi_\lambda^{-1}[\phi_\lambda x, \phi_\lambda y]_{\mathcal{G}} = [x, y]_{\mathcal{G}}, \quad (4.8)$$

for all $x, y \in U$.

Proposition 4.3.1 *Suppose $\mathcal{G} = (U, [,])$ is a Lie algebra, with ϕ_λ as above with $\lambda > 0$. Then $\mathcal{G}_\lambda = (U, [,]_\lambda)$, where $[x, y]_\lambda = \phi_\lambda^{-1}[\phi_\lambda x, \phi_\lambda y]$, is a Lie algebra isomorphic to \mathcal{G} .*

Proof Since ϕ_λ is invertible,

$$[x, y]_\lambda = \phi_\lambda^{-1}[\phi_\lambda x, \phi_\lambda y] \quad (4.9)$$

is equivalent to

$$\phi_\lambda[x, y]_\lambda = [\phi_\lambda x, \phi_\lambda y], \quad (4.10)$$

and this certainly shows that \mathcal{G} is a homomorphic Lie algebra to \mathcal{G}_λ . Finally, ϕ being a member of $\mathcal{GL}(U)$ shows that it is one-to-one and onto.

Proposition 4.3.2 *The map $\lambda \mapsto [,]_\lambda$ from $(0, 1]$ to the space of alternating bilinear forms on U , to which the Lie brackets belong in particular, is a continuous map.*

Proof This is clear from seeing this map as a composition of the map ϕ_λ and the Lie bracket, both of which are continuous.

Proposition 4.3.3 *Suppose \mathcal{G} is a Lie algebra with underlying vector space U and $(\phi_\lambda)_{\lambda \in \mathbb{R}^+} \subset \mathcal{GL}(U)$ is such that*

$$[x, y]_1 = \lim_{\lambda \rightarrow 0} \phi_\lambda^{-1} [\phi_\lambda x, \phi_\lambda y] \quad (4.11)$$

exists. Then $[\cdot, \cdot]_1$ is a Lie bracket on U and the resulting Lie algebra \mathcal{G}_1 is a contraction of \mathcal{G} .

Proof We need to show that $[\cdot, \cdot]_1$ is a Lie bracket.

1. **Linearity.**

$$[aX + bY, Z]_1 = \lim_{\lambda \rightarrow 0} [aX + bY, Z]_\lambda \quad (4.12)$$

$$= \lim_{\lambda \rightarrow 0} a[X, Z]_\lambda + \lim_{\lambda \rightarrow 0} b[Y, Z]_\lambda \quad (4.13)$$

$$= a[X, Z]_1 + b[Y, Z]_1. \quad (4.14)$$

2. **Antisymmetry**

$$[X, Y]_1 = \lim_{\lambda \rightarrow 0} [X, Y]_\lambda \quad (4.15)$$

$$= \lim_{\lambda \rightarrow 0} -[Y, X] \quad (4.16)$$

$$= -[Y, X]_1. \quad (4.17)$$

3. Jacobi Identity

$$[[X, Y]_1, Z]_1 + [[Y, Z]_1, X]_1 + [[Z, X]_1, Y]_1 = \quad (4.18)$$

$$\lim_{\lambda \rightarrow 0} \{ [[X, Y]_\lambda, Z]_\lambda + [[Y, Z]_\lambda, X]_\lambda + [[X, Y]_\lambda, Z]_\lambda \} = \quad (4.19)$$

$$0 \quad (4.20)$$

Proposition 4.3.4 (Inonu-Wigner contraction) *Let \mathcal{G} be a Lie algebra with underlying vector space U , and let \mathfrak{t} be a Lie subalgebra. Let V denote a subspace of U complementary to \mathfrak{t} . Thus $x \in U$ can be uniquely written*

$$x = x_{\mathfrak{t}} + x_V. \quad (4.21)$$

For $\lambda \in \mathcal{R}^+$, define a map $\phi_\lambda \in \mathcal{GL}(U)$ by

$$\phi_\lambda(x) = x_{\mathfrak{t}} + \lambda x_V. \quad (4.22)$$

Let x, y be in U . then $[\lim_{\lambda \rightarrow 0} \phi_\lambda^{-1} [\phi_\lambda x, \phi_\lambda y]]$ exists and is equal to

$$[x, y]_{\mathfrak{t}} = [x_{\mathfrak{t}}, y_{\mathfrak{t}}] + [x_{\mathfrak{t}}, y_V]_V + [x_V, y_{\mathfrak{t}}]_V. \quad (4.23)$$

Proof

$$[x, y]_1 = \lim_{\lambda \rightarrow 0} \phi_\lambda^{-1} [\phi_\lambda x, \phi_\lambda y] \quad (4.24)$$

$$= \lim_{\lambda \rightarrow 0} \phi_\lambda^{-1} [\phi_\lambda (x_\dagger + x_v), \phi_\lambda (y_\dagger + y_v)] \quad (4.25)$$

$$= \lim_{\lambda \rightarrow 0} \phi_\lambda^{-1} [x_\dagger + \lambda x_v, y_\dagger + \lambda y_v] \quad (4.26)$$

$$= \lim_{\lambda \rightarrow 0} \phi_\lambda^{-1} ([x_\dagger, y_\dagger] + \lambda [x_v, y_\dagger] + \lambda [x_\dagger, y_v] + \lambda^2 [y_v, y_v]) \quad (4.27)$$

But \dagger is a Lie subalgebra, so $[x_\dagger, y_\dagger] \in \dagger$ and so:

$$[x, y]_1 = \lim_{\lambda \rightarrow 0} \phi_\lambda^{-1} ([x_\dagger, y_\dagger] + \lambda ([x_v, y_\dagger]_\dagger + [x_v, y_\dagger]_v)) \quad (4.28)$$

$$+ \lambda ([x_\dagger, y_v]_\dagger + [x_\dagger, y_v]_v) + \lambda^2 ([x_v, y_v]_\dagger + [y_v, y_v]_v)) \quad (4.29)$$

$$= [x_\dagger, y_\dagger] + [x_v, y_\dagger]_v + [x_\dagger, y_v]_v \quad (4.30)$$

Note that for the limit to exist, \dagger has to be a Lie subalgebra. Otherwise,

$$\phi_\lambda^{-1}([x_\dagger, y_\dagger]) = \phi_\lambda^{-1}([x_\dagger, y_\dagger]_\dagger + [x_\dagger, y_\dagger]_v) \quad (4.31)$$

$$= [x_\dagger, y_\dagger]_\dagger + \frac{1}{\lambda} [x_\dagger, y_\dagger], \quad (4.32)$$

showing that when λ tends to 0, the limit does not exist.

Thus $[\cdot, \cdot]_1$ is a Lie bracket on U , and the resulting Lie algebra is a contraction of \mathcal{G} . A contraction arising this way is called an *Inonu-Wigner contraction with respect to \dagger* .

Definition 4.3.2 *Let G be a Lie group and K a Lie subgroup, reductive in G . Then \mathcal{G} may be written as $\mathfrak{t} \oplus V$, where the decomposition is Ad_K -invariant ($Ad_K V \subseteq V$). Let $V \rtimes K$ denote the semidirect product of V by K relative to this action. Define $\pi_\lambda : V \times K \rightarrow G$ by $\pi_\lambda(vk) = \exp_G(\lambda v)k$ for each $\lambda \in \mathcal{R}^+$.*

Proposition 4.3.5 *The Lie algebra of $V \rtimes K$ is precisely \mathcal{G} of proposition 4.3.3. And the differential of π_λ at the identity is precisely the map ϕ_λ of definition 4.3.1.*

Proof The first statement follows from the fact that the decomposition is Ad_K -invariant. The second follows from applying the definition of differential of a smooth map.

Thus we can define the contractions of on a group level as:

Definition 4.3.3 *The semidirect product $V \rtimes K$ is called a contraction of G with respect to K , and the family $(\pi_\lambda)_{\lambda \in \mathcal{R}^+}$ of maps $V \rtimes K \rightarrow G$ is called the family of contraction maps.*

In practice, once we obtain a contraction on the Lie algebra level, we derive a contraction of the Lie group level simply by exponentiating the basis involved. We do not know of any general method of contracting their coefficients.

4.4 Contractions of the Affine Group to the Heisenberg Group

4.4.1 Contractions on the Lie algebra Level

We now focus on $G_A = \mathcal{A}^1 \times \mathcal{R}$, the group of 3×3 matrices of the form:

$$\begin{pmatrix} a & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^x \end{pmatrix} \quad (4.33)$$

where $a, b, x \in \mathcal{R}$. The group $G_{\mathcal{A}}$ is essentially the affine group, however, crossed with the real line, which acts as its center.

Since $G_{\mathcal{A}}$ is a matrix group, its Lie algebra $\mathcal{G}_{\mathcal{A}}$ can be realized as a matrix algebra. A set of basis elements for $\mathcal{G}_{\mathcal{A}}$ consists of:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad (4.34)$$

with commutation relations:

$$[A_1, A_2] = A_2, [A_1, A_3] = [A_2, A_3] = \Theta, \quad (4.35)$$

where Θ is the zero matrix.

We first contract $G_{\mathcal{A}}$ to \mathcal{H} in a coordinate free context. Let \dagger be the subalgebra spanned by $\{C_2 = 2(A_2 - A_3)\}$. A complementary subspace V is that space spanned by $\{C_3 = A_2 + A_3, C_1 = A_1\}$, so that ϕ_{λ} is the map:

$$a_1 C_1 + a_2 C_2 + a_3 C_3 \longmapsto \lambda a_1 C_1 + a_2 C_2 + \lambda a_3 C_3. \quad (4.36)$$

Calculating,

$$\lim_{\lambda \rightarrow 0} \phi_{\lambda}^{-1}[\phi_{\lambda} C_1, \phi_{\lambda} C_2] = \lim_{\lambda \rightarrow 0} \phi_{\lambda}^{-1}[\phi_{\lambda} A_1, 2(A_2 - A_3)] \quad (4.37)$$

$$= \lim_{\lambda \rightarrow 0} \phi_{\lambda}^{-1}[\lambda A_1, 2A_2] \quad (4.38)$$

$$= \lim_{\lambda \rightarrow 0} \phi_{\lambda}^{-1}(2\lambda)A_2 \quad (4.39)$$

$$= \lim_{\lambda \rightarrow 0} \phi_{\lambda}^{-1}2\lambda\{(A_2 - A_3) + (A_2 + A_3)\} \quad (4.40)$$

$$= \lim_{\lambda \rightarrow 0} \phi_{\lambda}^{-1}\lambda(C_2 + C_3) \quad (4.41)$$

$$= \lim_{\lambda \rightarrow 0} 2\lambda(C_2 + \frac{1}{\lambda}C_3) \quad (4.42)$$

$$= 2C_3. \quad (4.43)$$

Hence: $[C_1, C_2]_1 = C_3$. All the other structure constants can be verified in the same manner to be 0. Therefore the three dimensional Lie algebra contracts to the Heisenberg Lie algebra.

We now follow Kalnins and Miller and repeat the above contraction in coordinates. We choose a new path of basis elements parametrized by ϵ :

$$C_1(\epsilon) = \epsilon A_1 = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.44)$$

$$C_2(\epsilon) = A_2 - A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (4.45)$$

$$C_3(\epsilon) = \epsilon A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \quad (4.46)$$

It may be verified that:

$$\begin{aligned} [C_1(\epsilon), C_2(\epsilon)] &= \epsilon C_2(\epsilon) + C_3(\epsilon); \\ [C_2(\epsilon), C_3(\epsilon)] &= [C_1(\epsilon), C_2(\epsilon)] \\ &= \Theta. \end{aligned} \quad (4.47)$$

Letting $\epsilon \longrightarrow 0$, we obtain a new Lie Algebra. whose basis has commutation relations:

$$\begin{aligned} [C_1(0), C_2(0)] &= C_3(0); \\ [C_2(0), C_3(0)] &= [C_1(0), C_2(0)] \\ &= \Theta. \end{aligned} \quad (4.48)$$

Again, we observe that the affine Lie algebra contracts to the Heisenberg

$$\begin{array}{ccccc}
 \mathbf{R}^3 & \times & \mathbf{R}^3 & \longrightarrow & \mathbf{R}^3 \\
 \downarrow & & \psi \times \psi & & \downarrow \psi \\
 G_A^\epsilon & \times & G_A^\epsilon & \longrightarrow & G_A^\epsilon
 \end{array}$$

Figure 4.1: Calculation of new operation of the general affine group.

Lie algebra. The coordinate approach will enable us to work with the affine Lie group as the next section will show.

4.4.2 Contractions on the Lie Group Level

We will now exponentiate \mathcal{G}_a^ϵ to obtain the sequence of the corresponding Lie groups G_A^ϵ . Let X be a member of \mathcal{G}_a^ϵ . Since X can be expressed as $X = -\beta\mathcal{C}_1 + a\mathcal{C}_2 + \alpha\mathcal{C}_3$, we obtain:

$$\exp\left(\begin{pmatrix} -\beta\epsilon & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\epsilon c - a} \end{pmatrix}\right) = \begin{pmatrix} e^{-\beta\epsilon} & \frac{a}{\beta\epsilon}[e^{-\beta\epsilon} - 1] & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\epsilon c - a} \end{pmatrix} \quad (4.49)$$

This exponentiation of the Lie algebra yields a continuum of groups parametrized by ϵ . The result is a new coordinatization of G_A^ϵ associated to the ϵ -Lie algebra. We would like to know what the multiplication rule becomes in these coordinates, so that we can find the contracted rule when $\epsilon \rightarrow 0$. The figure 4.1 related to 3.1 illustrates the process:

The map $\phi : \mathbf{R}^3 \rightarrow \exp(\mathcal{G}_a^\epsilon)$ is the map:

$$(a, \beta, c) \mapsto \begin{pmatrix} e^{-\beta c} & \frac{a}{\beta c} [e^{-\beta c} - 1] & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\epsilon c - a} \end{pmatrix}. \quad (4.50)$$

By considering the Jacobian, since ϕ can also be seen as a map from \mathbf{R}^3 to \mathbf{R}^3 , and applying L'Hopital's rule, we can see that ϕ is a one-to-one differentiable transformation. Hence its inverse exists:

$$\begin{aligned} \phi^{-1} : \exp(\mathcal{G}_a^\epsilon) &\rightarrow \mathbf{R}^3 \\ \begin{pmatrix} x & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix} &\mapsto \left(\frac{y \ln(x)}{x-1}, -\frac{1}{\epsilon} \ln(x), \frac{1}{\epsilon} [\ln(z) + \frac{y \ln(x)}{x-1}] \right). \end{aligned} \quad (4.51)$$

The operation, \star , of the group $\exp(\mathcal{G}_a^\epsilon)$ is simply the usual matrix multiplication.

By seeing ϕ as a change in coordinates, we find the corresponding group operation on \mathbf{R}^3 as follows:

$$\begin{aligned} (a, \beta, c) \star (a', \beta', c') &= \\ \left((e^{-\beta c} \frac{a'}{\beta' \epsilon} [1 - e^{-\beta' \epsilon}] + \frac{a}{\beta \epsilon} [e^{-\beta c} - 1]) \frac{(\beta + \beta') \epsilon}{(1 - e^{-(\beta + \beta') \epsilon})}, \right. \\ &\quad \left. \beta + \beta', \right. \\ c + c' - \left(\frac{a}{\epsilon} - \left[\frac{a}{\beta \epsilon} [e^{-\beta c} - 1] \right] \frac{(\beta + \beta')}{e^{-(\beta + \beta') \epsilon} - 1} \right) & \end{aligned}$$

$$- \left(\frac{a'}{\epsilon} + \left[\frac{a'}{\beta' \epsilon} [1 - e^{-\beta' \epsilon}] \right] \frac{(\beta + \beta') e^{-\beta \epsilon}}{e^{-(\beta + \beta')} - 1} \right). \quad (4.52)$$

Now let $\epsilon \rightarrow 0$. By expanding the exponential functions as power series, we obtain the contracted multiplication rule:

$$(a, \beta, c) \star (a', \beta', c') = (a + a', \beta + \beta', c + c' + \frac{a\beta' - a'\beta}{2}) \quad (4.53)$$

This group operation is identical to the Heisenberg group under symplectic coordinates. Therefore we have contracted the affine group to the Heisenberg group.

It is instructive to consider the groups parametrized by ϵ . We found that the group $\exp(\mathcal{G}_a^1)$ is isomorphic to each $\exp(\mathcal{G}_a^\epsilon)$ by an isomorphism ψ :

$$(a, \beta, c) \longmapsto \left(\frac{a}{\epsilon}, \epsilon \beta, c \right). \quad (4.54)$$

This fact is easily verified by a routine calculation. We will use this isomorphism in the next chapter.

4.4.3 Contractions on the Level of Group Representations

The group G_A has a unitary representation on $L^2(\mathcal{R})$ parametrized by λ as follows 3.3.1:

$$U^\lambda(a, b, c)g(t) = a^{-\frac{1}{2}} e^{2\pi i \lambda c} g\left(\frac{t+b}{a}\right). \quad (4.55)$$

In the previous sections, we always assumed implicitly that $\lambda = 1$, however, as we will see, the role of λ becomes more important when contracting group representations.

To contract, we need a representation K^λ equivalent to U^λ above, obtained by intertwining U^λ by $e^{2\pi i \lambda t}$:

$$K^\lambda = e^{-2\pi i \lambda t} U^\lambda(a, b, c) e^{2\pi i \lambda t}, \quad (4.56)$$

noting that any $f \in L^2(\mathcal{R})$ may be expressed as $g(t) = e^{2\pi i \lambda t} h(t)$ for some $h(t) \in L^2(\mathcal{R})$. Thus:

$$K^\lambda(a, b, c)g(t) = a^{-\frac{1}{2}} e^{-2\pi i \lambda c} e^{2\pi i \lambda \frac{t(1-a)+b}{a}} g\left(\frac{t+b}{a}\right). \quad (4.57)$$

Now we combine the parametrization ϕ of G'_A in equation 4.50 with the representation K^λ with $\lambda = \frac{1}{\epsilon}$. Thus $K^{\frac{1}{\epsilon}} \circ \phi$ has the form:

$$K^{\frac{1}{\epsilon}}(e^{-\beta\epsilon}, \frac{a}{\beta\epsilon}[e^{-\beta\epsilon} - 1], \epsilon c - a)g(t) = \quad (4.58)$$

$$e^{-\frac{\beta\epsilon}{2}} e^{-2\pi i c} e^{2\pi i \left[\left(\frac{e^{\beta\epsilon} - 1}{\epsilon} \right) t + a \left(\frac{\beta\epsilon - e^{\beta\epsilon} + 1}{\beta\epsilon^2} \right) \right]} g(e^{\beta\epsilon} t + \frac{a}{\beta\epsilon}[1 - e^{\beta\epsilon}]) \quad (4.59)$$

The process of letting $\epsilon \rightarrow 0$ is so similar to the other cases that we will simply state the result: after contraction, we obtain a new representation $K^0[a, \beta, c]$:

$$K^0[a, \beta, c]g(t) = e^{\pi i(2c - a\beta)} e^{2\pi i\beta t} g(t + a). \quad (4.60)$$

Thus we have obtained a representation of the Heisenberg group, which is what we wanted.

4.4.4 Contractions and Ambiguity Functions

We now contract the ambiguity functions themselves. This involves an additional "trick" as will be seen in this section.

As shown earlier 3.3.1, the wideband ambiguity function is a scaled coefficient of the group representation $U^\lambda(a, b, c)$:

$$W_{f,g}^\lambda(a, b) = \langle f(t), e^{-2\pi i\lambda c} U^\lambda(a, b, c)g(t) \rangle \quad (4.61)$$

$$= a^{\frac{1}{2}} \int_{-\infty}^{\infty} f(t) g\left(\frac{t-b}{a}\right) dt \quad (4.62)$$

However, to create our sequence of wideband ambiguity functions to converge to the narrowband ambiguity function, we use the functions f^ϵ and g^ϵ , where $f^\epsilon = e^{\frac{2\pi i\epsilon}{c}} f$ and similarly with g . We also use the ϕ coordinates for the argument of U and choose $\lambda = \frac{1}{\epsilon}$, and premultiply the ambiguity function by an appropriate factor:

$$e^{-2\pi i(\frac{\epsilon c - a}{\epsilon})} W_{f^\epsilon, g^\epsilon}(e^{-\beta\epsilon}, \frac{a}{\beta\epsilon}[e^{-\beta\epsilon} - 1], \epsilon c - a) = \quad (4.63)$$

$$< e^{\frac{2\pi i a}{\epsilon}} f, U^{\frac{1}{\epsilon}}(e^{-\beta\epsilon}, \frac{a}{\beta\epsilon}[e^{-\beta\epsilon} - 1], \epsilon c - a) e^{\frac{2\pi i a}{\epsilon}} g > = \quad (4.64)$$

$$< f, e^{\frac{2\pi i a}{\epsilon}} U^{\frac{1}{\epsilon}}(e^{-\beta\epsilon}, \frac{a}{\beta\epsilon}[e^{-\beta\epsilon} - 1], \epsilon c - a) e^{\frac{2\pi i a}{\epsilon}} g^\epsilon > \quad (4.65)$$

But the right hand side of the last bracket is just $K^{\frac{1}{\epsilon}}$ as in equation 4.56. So we continue:

$$e^{-2\pi i(\frac{\epsilon c - a}{\epsilon})} W_{f^\epsilon, g^\epsilon}(e^{-\beta\epsilon}, \frac{a}{\beta\epsilon}[e^{-\beta\epsilon} - 1], \epsilon c - a) = \quad (4.66)$$

$$< f, K^{\frac{1}{\epsilon}}(e^{-\beta\epsilon}, \frac{a}{\beta\epsilon}[e^{-\beta\epsilon} - 1], \epsilon c - a) g > \quad (4.67)$$

Letting $\epsilon \rightarrow 0$, we obtain by the previous section:

$$< f, K^0(a, \beta, c) g > = \quad (4.68)$$

$$< f(t), e^{-2\pi i c} e^{2\pi i \beta t} e^{\pi i a \beta} g(t + a) > = \quad (4.69)$$

$$e^{2\pi i c} e^{\pi a \beta} \int_{-\infty}^{\infty} f(t) \overline{g(t + a)} e^{-2\pi i \beta t} dt = \quad (4.70)$$

$$e^{2\pi i c} e^{\pi a \beta} N_{f, g}(a, -\beta), \quad (4.71)$$

which is essentially the narrowband ambiguity function, up to phase. We have thus

contracted the coefficients of the two groups, approximating the narrowband ambiguity function by the narrowband via a formal mathematical apparatus, without any reference to physical factors.

Chapter 5

Volume and Contractions

5.1 Introduction

We would like to apply the idea of contractions to gain some insight into the important concept of ambiguity volume. After introducing ambiguity volume, we will explore the relationship between volume and invariant measures via the Schur-Godement- Frobenius theorem. We then will study how volume changes with contractions. Our aim is to make rigorous the statement of Sibul and Titlebaum [37, page 86]: that the volume of the wideband ambiguity function is asymptotically conserved as the narrowband case is approached.

5.2 Ambiguity Volume

The ambiguity functions of interest in this chapter are the *auto-ambiguity functions*, for example:

$$N_f(x, y) = \int_{-\infty}^{\infty} f(t) \overline{f(t+x)} e^{-2\pi i y} dt, \quad (5.1)$$

a special case of the cross-ambiguity function. These auto-ambiguity functions may be seen as functions of two variables on their groups. The surface over the $x - y$

plane for the narrowband functions $(x, y, N_f(x, y))$ called the *narrowband ambiguity surface*.

Ambiguity volume, V_N is simply the volume under the ambiguity surface squared:

$$V_N(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |N_f(x, y)|^2 dx dy. \quad (5.2)$$

Similarly, wideband ambiguity volume, $V_W(f)$ is defined to be:

$$V_W(f) = \int_0^{\infty} \int_{-\infty}^{\infty} |W_f(s, \tau)|^2 \frac{ds}{s} d\tau. \quad (5.3)$$

The interesting fact is that narrowband ambiguity volume is always conserved:

Proposition 5.2.1

$$V_N(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |N_f(x, y)|^2 dx dy = \|f\|^4, \quad (5.4)$$

that is, if $\|f\| = 1$, then:

$$V_N(f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |N_f(x, y)|^2 dx dy = 1. \quad (5.5)$$

Proof We reference the proof in [22, page 640]. It also will be a natural consequence of the Godement-Frobenius-Schur theorem to be stated later.

The proposition above has been called the “radar uncertainty principle”, or the “law of conservation of ambiguity”. It can be seen as saying that the ambiguity surface is like a water bed: to squeeze a desired small peak at a particular range and speed implies that other peaks will rise somewhere. One of the main problems of radar was to see what kind of surfaces were ambiguity surfaces. One of the approaches to the solution of this problem was in seeing what kind of invariance properties they exhibited, which motivated Cook and Bernfield [12, page 68]. The radar conservation property was found to be the “most important ambiguity function constraint since it implies that all signals are equally good (or bad) as long as they are not compared against a specific radar environment” [12, page 70]. However, the distribution of ambiguity does change from signal to signal. For example, a signal with short time duration would produce a peak that could resolve two objects flying close together, but conservation would force the objects velocities to be unresolvable. When coupled with the easily proven fact that the maximum of an autoambiguity function occurs at the origin, the choice of surfaces becomes even more limited.

Wideband ambiguity does not hold always, for:

Proposition 5.2.2 *Let $f \in L^2(\mathcal{R})$. Then:*

$$\int_0^\infty \int_{-\infty}^\infty |W_f(s, \tau)|^2 \frac{ds}{s} d\tau = \|f\|^2 \int_0^\infty |\hat{f}(\omega)|^2 \frac{d\omega}{\omega}. \quad (5.6)$$

Proof We again reference [22, page 643] for details.

There are certainly $f \in L^2(\mathcal{R})$ such that $\int_0^\infty |\hat{f}(\omega)|^2 \frac{d\omega}{\omega}$ is not defined. The loss of conservation in the wideband realm causes difficulty for selection of surfaces. Only one constraint in wideband signal design remains. An approach to classifying all wideband ambiguity surfaces is detailed in [1].

5.3 Volume and Group Theory

To apply group theory for insight into the volume problem, we need a few definitions.

Definition 5.3.1 *Let U be a unitary representation of a Lie group G on a Hilbert space H . Then U is said to be a square integrable representation if there is a vector $g \in H$ such that:*

$$\int_G |\langle U(x)g, g \rangle|^2 d\mu(x) < \infty. \quad (5.7)$$

Such a vector g is called admissible.

We now quote without proof the Froebenius-Schur-Godement theorem:

Proposition 5.3.1 *If U is a square integrable representation of a group G on a Hilbert space H , there exists a unique self-adjoint operator Q such that:*

- i) The set of admissible vectors coincide with the domain of Q .*
- ii) Let g_1 and g_2 be two admissible vectors and let f_1 and f_2 be any vectors in H . Then:*

$$\int_G \langle f_1, U(x)g_1 \rangle \overline{\langle f_2, U(x)g_2 \rangle} d\mu(x) = \langle Qg_2, Qg_1 \rangle \langle f_1, f_2 \rangle. \quad (5.8)$$

iii) Finally, if G is unimodular, then Q is a multiple of the identity operator; hence the domain of Q will be all of H , that is, all of H constitutes the set of admissible vectors for the representation.

The proposition is proved in detail in [38, page 62].

Now we will apply these new facts to our ambiguity functions. The equation 5.7 may be recast as stating that U is square-integrable if there is a signal g such that its ambiguity volume is finite. This general statement applies to both narrow and wideband when considering the appropriate representation as indicated previously. The equations 5.5 and 5.6 show that under this formulation, $U_{\mathcal{H}}$ and $U_{\mathcal{A}_2}$ are square integrable representations of their respective groups.

Since the Heisenberg group is unimodular [22, page 640], all of $L^2(\mathcal{R})$ is admissible by the Frobenius-Schur-Godement theorem. This is not true for the affine group: there are functions for which the equation 5.6 will not be finite on the right hand side. And the Frobenius-Schur-Godement theorem does not guarantee that all of $L^2(\mathcal{R})$ would be admissible for $U_{\mathcal{A}_1}$. By the converse of the third part of the theorem, we see that the affine group cannot be unimodular. And sure enough, it is not [22, page 643]. We can see that there is some connection between unimodularity of groups and ambiguity volume.

We wanted to see how invariant measures behaved in the ϵ -groups involved in the contraction of the affine group to the Heisenberg group.

Proposition 5.3.2 *The left invariant measure of an ϵ -group is*

$$\frac{1 - e^{-\beta\epsilon}}{\beta\epsilon} dad\beta dc \quad (5.9)$$

and the right:

$$\frac{e^{\beta\epsilon} - 1}{\beta\epsilon} dad\beta dc. \quad (5.10)$$

Proof We may prove the proposition by straightforward verification. For example, the right invariant measure may be verified by showing that:

$$\int f[(a, \beta, c) \cdot (a', b', c')] dad\beta dc = \quad (5.11)$$

$$\int f(u, v, w) \frac{\partial(a, \beta, c)}{\partial(u, v, w)} du dv dw = \quad (5.12)$$

$$\int f(u, v, w) du dv dw, \quad (5.13)$$

where u , v , and w denote the components of the right hand side of equation 4.52 in their respective orders, and $\frac{\partial(a, \beta, c)}{\partial(u, v, w)}$ denotes the Jacobian of the transformation, computed by taking the reciprocal of $\frac{\partial(u, v, w)}{\partial(a, \beta, c)}$.

An interesting aside is that the isomorphism ψ from the previous chapter, carries the left invariant measure of the group when $\epsilon = 1$ to the left invariant measure of any specified ϵ -group.

As long as $\epsilon > 0$, we can see that the groups will never be unimodular, but as $\epsilon \rightarrow 0$, the groups become more and more unimodular. It seems that the Frobenius-Schur-Godement theorem might help us to see that ambiguity volume becomes conserved.

Thus the statement that the "wideband ambiguity volume is asymptotically conserved" now has a mathematical foundation.

Chapter 6

Conclusions and Questions for Further Research

We have shown how group theory can provide insight into some of some problems in RADAR and SONAR signal processing. We began by noting that the RADAR and SONAR ambiguity functions are coefficients of the unitary representations of the Heisenberg and affine groups respectively. This basic fact not only unifies the various forms of the wideband ambiguity function, but helps us in simplifying various computations. We also showed how contractions helped model the approximations of the wideband to the narrowband ambiguity function under suitable conditions. And we showed that the wideband ambiguity volume is not conserved, even when the affine group in question is very close to the Heisenberg group. Finally, we showed that harmonic analysis on finite groups may provide insight into creating another transform.

Our work is just the beginning of a "wideband" version of Schempp's work [38]. For example, the characterization of narrowband ambiguity surfaces was shown to be a natural consequence of harmonic analysis on the Heisenberg group by Schempp. He also solved the invariant problem, finding the group under whose motions the narrowband ambiguity surfaces remained invariant. A wideband counterpart to these results seems daunting, since the affine group is not nilpotent as

the Heisenberg, but only solvable. To complicate Schempp's approach further, the affine group is not unimodular. The theory of positive definite functions on the affine group, or the autoambiguity functions in this thesis, is not very well known for that reason.

We also would have liked to characterize the range of the wideband autoambiguity function as Auslander and Tolimieri did with the narrowband autoambiguity function [4]. However, the affine group seems immune to their approach, since it lacks a discrete subgroup to quotient it to a compact manifold, to enable us to apply some theory of "solv-theta" functions to it as they did.

Many questions remain to be answered. For example, what is the geometric picture of this contraction? Dooley shows some elegant geometric pictures for some of semisimple group contractions [15]. Can Kirillov coadjoint orbital theory be used to show what is happening with our two groups? Here, the difficulty is that the groups concerned lack "semisimplicity", and are solvable. We have calculated the coadjoint action of the ϵ -groups, and the action seems to "blow up" at the limit. What this means we are not sure. Is the relationship between wideband and narrowband analogous to the relationship between quantum and classical mechanics? Is there a method of "quantization" from a signal processing point of view?

The discrete affine group harmonic analysis also provides many interesting questions. Is there a finite cross ambiguity function? How should the signals be

sampled? Is there a solvable analog to Shannon's sampling theorem? What is the significance of the affine Fourier transform? The affine group here seems to lack the beautiful "Stone-von Newman" property of the Heisenberg: that the Heisenberg has essentially one unitary representation.

Finally, we hope to see more areas of signal processing "invaded" by modern mathematics. While the insights gained would be invaluable, perhaps new devices to serve the needs of humanity will be created as a result.

Appendix

The Discrete Affine Fourier Transform

A.1 INTRODUCTION

Discrete Fourier Transforms (DFT) have been standard tools in digital signal processing for approximately three decades. If we view the theory of the Discrete Fourier Transform as harmonic analysis on a group of integers modulo a prime number, we can gain new insights that will allow us to develop new discrete Fourier transforms as harmonic analysis on other discrete groups.

Group theory enables us to see various transforms in a unifying manner. For example, the Fourier transform is intimately related to the group of the real numbers; the Z-transform is connected with the integers; and the Discrete Fourier Transform is related to the group of integers mod n . We introduce a new transform by considering the finite affine group, and find that all the theory of the Discrete Fourier Transforms generalizes in a pleasant way. The formula for the Discrete Affine Transform remains essentially the same as the Discrete Fourier Transform.

The underlying theory for this transform, namely that of harmonic analysis on finite groups, has been known for quite some time [23, 24]. We have essentially applied the theory to the finite affine group, having been motivated by radar ambiguity function theory [36, 9, 5].

We will review familiar properties of DFT's that are consequences of harmonic analysis on finite fields, and then investigate harmonic analysis on the *finite* affine group \mathcal{A}_p , where p is a prime number. Focusing on a specific finite affine group, we then develop the Discrete Affine Fourier Transform (DAFT), using a specific finite affine group for illustration. (This transform should not be confused with the Affine Wavelet Transform). After establishing the Inverse Discrete Affine Fourier Transform (IDAFT), some transform properties analogous to the DFT will be shown, namely those involving the relationships between the DAFT and the norms and convolutions of its function spaces. Finally, we show how the DAFT would be useful for solving wideband inverse problems.

A.2 Some More Mathematical Background

Associated with a set S is the vector space of all complex valued functions on S denoted by $\mathcal{F}(S)$. This $\mathcal{F}(S)$ is a vector space in a natural way. If S is finite, $\mathcal{F}(S)$ is isomorphic to \mathcal{R}^n where n is the number of objects in S .

As an example, $\mathcal{F}(\mathcal{Z}_n)$ is the space of all complex valued sequences on $\mathcal{Z}_n: \{f | f = (f_0, \dots, f_{n-1})\}$. Addition on $\mathcal{F}(\mathcal{Z}_n)$ is simply componentwise addition of sequences and scalar multiplication is defined similarly. A *norm* may be defined on $\mathcal{F}(\mathcal{Z}_n)$: $\|f\|^2 = \sum_{k=0}^{n-1} f_k \bar{f}_k$. These norms induce an *inner product* as in usual Hilbert space theory.

The space $\mathcal{F}(\hat{G})$ is defined analogously, with \hat{G} forming the index set of the sequences. Each member ρ_k of the sequence will be a matrix of fixed dimension

n_k . A norm for $\hat{f} = (\hat{f}_1, \dots, \hat{f}_m)$ is: $\|\hat{f}\|^2 = \sum_{k=0}^m n_k \text{tr}(\hat{f} \hat{f}^*)$, where $*$ denotes the Hermitian transpose of a matrix. In the case of $\hat{\mathcal{Z}}_n$, when all the irreducible representations are one dimensional, the norm becomes the familiar one. Of course, these norms induce their corresponding inner products.

A.3 The Discrete Fourier Transform

As stated above, the theory of the Discrete Fourier Transform (DFT) is merely harmonic analysis on \mathcal{Z}_n , the group of integers modulo an integer [28, pages 546-548]. Harmonic analysis simply means expressing the functions on the group as a sum of the irreducible representations of the group.

The DFT can be seen as the mapping from the space of all complex sequences on \mathcal{Z}_n to the space of complex sequences on $\hat{\mathcal{Z}}_n$. Its explicit form is:

$$\hat{f}_\omega = \sum_{t=0}^{n-1} f_t e^{-\frac{2\pi i \omega t}{n}} \quad (\text{A.1})$$

The DFT formula is thus intimately related to the "harmonics" of \mathcal{Z}_n .

In addition, the Inverse Discrete Fourier Transform (IDFT) may be defined, also involving the characters of \mathcal{Z}_n :

$$f_t = \frac{1}{p} \sum_{\omega=0}^{n-1} \hat{f}_\omega e^{\frac{2\pi i \omega t}{n}}. \quad (\text{A.2})$$

The DFT has some nice properties. It is well known that the DFT preserves norms: $\|f\|^2 = \frac{1}{n} \|\hat{f}\|^2$. Secondly, the (cyclic) *convolution* of two n -length sequences

defined by:

$$(f * g)_t = \sum_{\tau=1}^{n-1} f_{\tau} g_{t-\tau}, \quad (\text{A.3})$$

$t - \tau$ performed with modulo n arithmetic, is conveniently mapped to component-wise multiplication of their transforms: $(\widehat{f * g})_{\omega} = \hat{f}_{\omega} \cdot \hat{g}_{\omega}$. These two properties of the DFT form a foundation for a full range of applications [33].

A.4 The Discrete Affine Fourier Transform

A.4.1 The Finite Affine Group and Its Representation Theory

Motivated by the affine group's intimate relationship to the wideband ambiguity function [36, 9], and inspired by Auslander and Tolimieri's approach to the DFT [5, page 866], we investigate the so called *finite affine group*, denoted by \mathcal{A}_p . This is the set of 2 by 2 matrices of the form:

$$\begin{pmatrix} s & \tau \\ 0 & 1 \end{pmatrix}, \quad (\text{A.4})$$

with s taking nonzero values in \mathcal{Z}_p and τ , any values in \mathcal{Z}_p . The group operation will be the usual matrix multiplication, however using modulo p arithmetic. Modulo p arithmetic is necessary for every nonzero integer to have a multiplicative inverse. This condition may be loosened slightly by requiring that the multiplicative group of the ring have a primitive generator, a condition true for \mathcal{Z}_n with $n = 2, 4, p^{\alpha}, 2p^{\alpha}$ [25, page 44]. However, for simplicity's sake, we shall consider only the case of \mathcal{A}_p .

A.4.2 The Discrete Affine Transform

The theory of Fourier transforms for the finite noncommutative group had been discovered for quite some time [24, page 114] . We will now construct the Discrete Affine Wavelet Transform (DAFT), the affine analogue of the DFT, using this theory. This process involves identifying “harmonics” on the group \mathcal{A}_p , and then applying the formula for the Fourier transform in the literature to this case [24, pages 77,81]. We shall use \mathcal{A}_3 as an illustrative example.

Identifying all the harmonics of \mathcal{A}_p .

We identified all the irreducible representations of $\hat{\mathcal{A}}_p$ in a semi *ad hoc* manner, by using two results in finite group representation theory. The first is that the number of the characters of a group is equal to the number of its conjugacy classes, and the second is that the sum of the squares of the dimensions of each irreducible representation is equal to the number of elements of the group [29]. We found that there were p conjugacy classes in \mathcal{A}_p , hence p irreducible representations.

And then:

$$\underbrace{1^2 + \dots + 1^2}_{p-1 \text{ times}} + (p-1)^2 = p(p-1). \quad (\text{A.5})$$

Therefore \mathcal{A}_p had to have p one dimensional characters and one $p-1$ -dimensional character. We found the explicit form of the representations by some trial and error, by consideration of their subgroups with some trial and error.

Incidentally, we found [34], which confirmed some of the results of this section. The explicit form of the representations follow. We note that not all of them are one dimensional.

The "harmonics" of \mathcal{A}_p consists of $p - 1$ one dimensional representations and one $p - 1$ -dimensional representation, a total of p elements, as follows:

$$\begin{aligned}
 \rho_0(g^s, \tau) &= 1 \\
 \rho_1(g^s, \tau) &= e^{\frac{2\pi i s}{p-1}} \\
 &\vdots = \\
 \rho_k(g^s, \tau) &= e^{\frac{2\pi i k s}{p-1}} \\
 &\vdots = \\
 \rho_{p-2}(g^s, \tau) &= e^{\frac{2\pi i (p-2)s}{p-1}}
 \end{aligned} \tag{A.6}$$

where g is the number with which every nonzero element of \mathbb{Z}_p may be expressed as a power of g . The $p - 1$ -dimensional representation ρ_{p-1} is that representation induced from the one dimensional representation:

$$\chi(1, \tau) = e^{\frac{2\pi i \tau}{p}} \tag{A.7}$$

The difficulty of expressing the DAFT explicitly in general results from not yet knowing the explicit form of this representation.

We now consider the representation theory of \mathcal{A}_3 . The group \mathcal{A}_3 consists of six elements, since there are two choices for s and three choices for τ . Because s is never zero, we can express it as a power of 2. (This fact holds for general \mathcal{A}_p , although the base of the exponential may not necessarily be 2). Hence the space of complex functions on the group are those sequences of length six. However, the set of "harmonics", the irreducible complex representations of \mathcal{A}_3 , consists of only three elements, a consequence of the group's noncommutativity. Hence we no longer have a group of harmonics, but something more complicated.

Explicitly, the characters of \mathcal{A}_3 are:

$$\rho_0(2^s, \tau) = 1 \quad (\text{A.8})$$

$$\rho_1(2^s, \tau) = (-1)^s \quad (\text{A.9})$$

$$\rho_2(2^s, \tau) = \begin{pmatrix} \omega^\tau & 0 \\ 0 & \omega^{2\tau} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^s \quad (\text{A.10})$$

Constructing the DAFT

The next two sections consist of applying the results of Hewitt and Ross [23, 24] to our specific group, involving translation of their mathematical formulations.

Given a sequence f on \mathcal{A}_p , that is, f has the form:

$$f = (f_{1,0}, \dots, f_{1,p}, f_{2,0}, \dots, f_{p-1,p}), \quad (\text{A.11})$$

we can construct a sequence \hat{f} on $\hat{\mathcal{A}}_p$, the set of "harmonics" of \mathcal{A}_p , called

the Discrete Fourier Transform (DAFT) as follows:

Since each representation of \mathcal{A}_p is associated to a finite linear space, we can use the space's inner product to define a *coefficient* of two vectors, u and v , associated with the representation:

$$C_{u,v}^{\rho_k}(g^s, \tau) = \langle \rho(g^s, \tau)u, v \rangle. \quad (A.12)$$

We use this concept to clarify the transform for a finite, noncommutative group such as ours. The DAFT is simply the sum:

$$\langle \hat{f}(\rho_k)u, v \rangle = \sum_{(g^s, \tau)} C_{u,v}^{\rho_k} f(g^s, \tau) \quad (A.13)$$

For \mathcal{A}_p , $\hat{f}(\rho_k)$ can be seen as an $n \times n$ matrix, where n is the dimension of the linear space involved.

The DAFT may be written explicitly for \mathcal{A}_3 as follows: if $f = (f_{10}, f_{11}, f_{12}, f_{20}, f_{21}, f_{22})$ is a sequence on \mathcal{A}_3 , the sequence \hat{f} consists of:

$$\begin{aligned} \hat{f}_{\rho_0} &= \sum_{(2^s, \tau) \in \mathcal{A}_p} f_{2^s, \tau} \\ \hat{f}_{\rho_1} &= \sum_{(2^s, \tau) \in \mathcal{A}_p} (-1)^s f_{2^s, \tau} \\ \hat{f}_{\rho_2} &= \begin{pmatrix} f_{10} + \omega f_{11} + \omega^2 f_{12} & f_{20} + \omega^2 f_{21} + \omega f_{22} \\ f_{20} + \omega f_{21} + \omega^2 f_{22} & f_{10} + \omega^2 f_{11} + \omega f_{12} \end{pmatrix} \end{aligned}$$

Constructing the IDAFT

The DAFT is not onto for general groups. However, since \mathcal{A}_p is finite, the DAFT is onto, and we have the existence of the Inverse Discrete Affine Fourier Transform [24, page 420]. Consider the space of all matrix valued sequences on $\hat{\mathcal{A}}_p$, that is, the space of all sequences with the "harmonics" of \mathcal{A}_p as the index set:

$$\hat{f} = (\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{p-1}), \quad (\text{A.14})$$

where the first $p-1$ entries are simply scalars, and the last entry is a $p-1 \times p-1$ matrix. The function f whose DAFT is \hat{f} is:

$$\begin{aligned} f_{g^s, \tau} = & \sum_{k=0}^{p-1} e^{\frac{2\pi i k \tau}{p-1}} \hat{f}_k \\ & + (p-1) \text{tr}(\hat{f}_{p-1} \cdot \rho_{p-1}(g^s, \tau)), \end{aligned} \quad (\text{A.15})$$

where tr denotes the trace of a matrix. This formula will be called the Inverse Discrete Affine Fourier Transform (IDAFT).

The IDAFT can be written for the case of \mathcal{A}_3 as follows: Let $\hat{f} = (\hat{f}_0, \hat{f}_1, \hat{f}_2)$ be a sequence on the set of "harmonics" of \mathcal{A}_3 . We note in passing that \hat{f}_2 is a two by two complex matrix. Then the function whose DAFT is \hat{f} is:

$$f_{2^s, \tau} = \hat{f}_0 + (-1)^s \hat{f}_1 + 2 \text{tr}(\hat{f}_2 \cdot \rho_2(2^s, \tau)) \quad (\text{A.16})$$

A.4.3 Interesting Properties of the DAFT

The DAFT has a number of interesting properties, quite similar to the DFT. We will mention two here, one related to norms and the other related to convolutions. We have proved them explicitly for \mathcal{A}_3 only, however the generalizations seem straight forward. Since the proofs are computational in nature, they have been omitted.

Defining a norm on the sequences on $\hat{\mathcal{A}}_3$ by:

$\|\hat{f}\|^2 = |\hat{f}_0|^2 + |\hat{f}_1|^2 + 2\text{tr}(\hat{f}_3\hat{f}_3^*)$, where $*$ denotes the Hermitian transpose, we found that: $\|\hat{f}\| = 6\|f\|^2$. This fact should generalize to be: $\|\hat{f}\| = p(p-1)a'\|f\|^2$, for general \mathcal{A}_p .

The affine group theoretical convolution of the form:

$$f * g(s, \tau) = \sum_{(x, y) \in \mathcal{A}_p} f(x, y) g[(x, y)^{-1}(s, \tau)] \quad (\text{A.17})$$

is mapped via the DAFT to componentwise multiplication:

$$\widehat{f * g} = \hat{g} \cdot \hat{f}. \quad (\text{A.18})$$

The right side denotes componentwise multiplication, as in the DFT case, with the matrix components multiplied as regular matrices. We were surprised that the products reversed order under the transform, a fact proven for the case of \mathcal{A}_3 only (This fact has not been mentioned in any works that we have seen).

A.5 Towards An Application of the DAFT

This section will describe a possible application of the ideas discussed. It will be seen that it can only be a sketch at best, yet hopefully detailed enough to motivate further research.

One of the important applications of affine deconvolution is the solution of wideband inverse problems. Consider the received signal $r(t)$ given by:

$$r(t) = \int_0^\infty \int_{-\infty}^\infty e^{-\frac{a_0}{2}} \rho(s_0, \tau_0) f(e^{-s_0} t - \tau_0) d\tau_0 ds_0 \quad (\text{A.19})$$

where $\rho(s_0, \tau_0)$ is the complex scattering strength at delay τ_0 and dilation s_0 , and $f(t)$ is the transmitted probing signal. The time varying transmission or scattering channel causes the signal to be delayed by τ_0 and its time-scale dilated by $e^{-\frac{a_0}{2}}$. For narrowband signals the time dilation reduces to the well known Doppler shift. The inverse problem is to determine $\rho(s_0, \tau_0)$ from received signal $r(t)$ and from the known probing signal $f(t)$. This can also be called the identification of linear time-varying systems [46].

Let us consider processing of the received signal by the correlation receiver. The correlation receiver computes an inner product of received signal $r(t)$ and the processing waveform $g(t)$, which is time-delayed and scaled. If the received signal contains colored noise in addition to the scattered probing signal, the optimum processing waveform is not the dilated and delayed version of the probing signal. We also recognize that the correlation receiver output $l(s, \tau)$ is an affine wavelet

transform of $r(t)$ with respect to the processing waveform $g(t)$. Thus,

$$l(s, \tau) = W_g r \quad (\text{A.20})$$

$$= \langle r(t), g(e^{-s}t - \tau)e^{-s} \rangle \quad (\text{A.21})$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_0^{\infty} f(e^{-s_0}t - \tau_0)e^{-\frac{s_0}{2}} \right. \quad (\text{A.22})$$

$$\left. g^*(e^{-s}t - \tau)e^{-\frac{s}{2}} d\tau_0 ds_0 \right) dt \quad (\text{A.23})$$

Interchanging the order of integration, we have:

$$l(s, \tau) = \int_{-\infty}^{\infty} \int_0^{\infty} \left(\int_{-\infty}^{\infty} e^{-\frac{s_0}{2}} f(e^{-s_0}t - \tau_0) g^*(e^{-s}t - \tau)e^{-\frac{s}{2}} dt \right) d\tau_0 ds_0. \quad (\text{A.24})$$

We now recognize that the inner integrals are an inner product and we have:

$$l(s, \tau) = \int_{-\infty}^{\infty} \int_0^{\infty} \rho(s_0, \tau_0) \langle U(s_0, \tau_0)f(t), U(s, \tau)g(t) \rangle d\tau_0 ds_0, \quad (\text{A.25})$$

where $U(s_0, \tau_0)$ and $U(s, \tau)$ are unitary affine transformations [9]. Since $U(s_0, \tau_0)$ is unitary, its adjoint is equal to its inverse and we have:

$$l(s, \tau) = \int_{-\infty}^{\infty} \int_0^{\infty} \rho(s_0, \tau_0) \langle f(t), U^{-1}(s_0, \tau_0)U(s, \tau)g(t) \rangle d\tau_0 ds_0 \quad (\text{A.26})$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \rho(s_0, \tau_0) \quad (\text{A.27})$$

$$\left(\int_{-\infty}^{\infty} f(t) g^*(e^{-(s-s_0)}t + e^{-(s-s_0)}\tau_0 + \tau) e^{-\frac{s-s_0}{2}} dt \right) d\tau_0 ds_0 \quad (\text{A.28})$$

The inner integral is the wideband ambiguity function,

$WA_{f,g}(e^{-(s-s_0)}, \tau - e^{-(s-s_0)}\tau_0)$. Hence the most recent equation may be seen as a continuous convolution on the affine group [32, 6]:

$$l(s, \tau) = \rho \star WA_{f,g} \quad (\text{A.29})$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \rho(s_0, \tau_0) WA_{f,g}[(e^s, \tau)(e^{s_0}, \tau_0)^{-1}] d\tau_0 ds_0 \quad (\text{A.30})$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} \rho(s_0, \tau_0) WA_{f,g}(e^{-(s-s_0)}, \tau - e^{-(s-s_0)}\tau_0) d\tau_0 ds_0. \quad (\text{A.31})$$

Thus the solution of the wideband inverse problem is equivalent to deconvolution.

The solution of this inverse problem can be attacked in several ways. One could use Kahlil's formula as suggested by Napharst [32], or wavelet transform inversion as suggested by Heil and Walnut [22], Young [46], and Grossmann and Morlet [17]. Use of the DAFT to deconvolve a discrete scattering problem would be another approach for additional insight into the solution of the inverse problem. Many difficult computational and discretization problems remain to be worked out. We hope that the Discrete Affine Fourier Transform Theory may have other interesting applications to wideband time-varying system theory.

A.6 Looking Ahead

We have performed some harmonic analysis on a finite noncommutative group. Noncommutativity generally makes such analysis very difficult, but finiteness of the groups saves the day. The same strategy may be applied to other groups such as the Heisenberg group for instance. As the affine group underlies the wideband ambiguity radar theory, the Heisenberg group plays a major role in narrowband ambiguity radar theory. Deconvolution is a major problem there too.

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